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GROUPS WHOSE HOMOMORPHIC IMAGES HAVE A TRANSITIVE NORMALITY RELATION

BY

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ABSTRACT. A group G is a *T-group* if $H \lhd K \lhd G$ implies that $H \lhd G$, i.e. normality is transitive. A *just non-T-group* (JNT-group) is a group which is not a T-group but all of whose proper homomorphic images are T-groups. In this paper all soluble JNT-groups are classified; it turns out that these fall into nine distinct classes. In addition all soluble $JN\overline{T}$-groups and all finite $JN\overline{T}$-groups are determined; here a group G is a $\overline{T}$-group if $H \lhd K \lhd L \le G$ implies that $H \lhd L$. It is also shown that a finitely generated soluble group which is not a T-group has a finite homomorphic image which is not a T-group.

1. Introduction and statement of results.

(1.1) **Definitions.** If P is a group theoretical property, a *just non-P-group* is a group which is not a P-group but all of whose proper homomorphic images are P-groups; for brevity we shall call these *JNP-groups*. For example, when P is commutativity, soluble JNP-groups have been studied by Newman ([16] and [17]) and also by Rosati [21].

Denote by T the property that *normality is transitive*; thus a group G has T if $H \lhd K \lhd G$ always implies that $H \lhd G$. Here we are concerned with *JNT-groups*; these can alternatively be defined as the groups whose nonnormal subnormal subgroups are core-free and form a nonempty set.

The principal object of this paper is to classify the soluble JNT-groups. This will be done by dividing the soluble JNT-groups into nine types (and eleven subtypes). While the descriptions of the different types vary in both complexity and precision, a rather clear picture of the structure of a soluble JNT-group emerges.

Notation.

$\langle X_\lambda : \lambda \in \Lambda\rangle$: subgroup generated by the (subsets) X_λ, $\lambda \in \Lambda$.

X^Y: normal closure of X in Y, i.e., the subgroup generated by all conjugates $x^y = y^{-1}xy$, ($x \in X$, $y \in Y$).

$[X, Y] = [X, {}_1Y]$: commutator subgroup generated by all commutators $[x, y] = x^{-1}y^{-1}xy$ ($x \in X$, $y \in Y$).

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$[X,_{i+1}Y] = [[X;\ _iY], Y]$.

$C_H(X)$: centraliser of X in H.

$\zeta(G)$: centre of G.

R^*: multiplicative group of units of R, a ring with unity.

FG: group algebra of a group G over a field F.

$\overset{\Omega}{\simeq}$: isomorphism of Ω-operator groups.

X^+: additive subgroup of a ring generated by X.

(1.2) **The classification of soluble** JNT**-groups.** We shall now describe the nine types of groups which can occur.

I. The nonabelian nilpotent groups all of whose proper homomorphic images are abelian[2], with the exception of the quaternion group of order 8.

II. Let L be an abelian group of type 2^∞ with generators $a_1, a_2, \cdots$ and relations $a_1^2 = 1$ and $a_{i+1}^2 = a_i$. Let $D = \langle d \rangle$ be a group of order 2. Define G to be the group generated by the direct product $L \times D$ and a cyclic group $\langle t \rangle$ of order 2 or 8 where $a^t = a^{-1}$, $d^t = a_1 d$ and $t^2 = 1$ or $a_2 d$ for all $a \in L$.

III. Let L and D be as in II except that D may now have order 1. Let X be an extra-special 2-group generated by elements of order 2 and let C be the direct product of $L \times D$ and X in which $\langle a_1 \rangle$ and the centre of X are amalgamated. Choose an element σ from $\mathrm{Hom}(X/\langle a_1 \rangle, \langle a_1 \rangle)$ and let $\langle t \rangle$ be a cyclic group of order 2 or 8. Define G to be the group generated by C and $\langle t \rangle$ where for all $a \in L$ and $x \in X$,

$$a^t = a^{-1}, \quad d^t = a_1 d \quad (\text{if } d \neq 1), \quad x^t = x(x\langle a_1 \rangle)^\sigma,$$

and

$$t^2 = 1 \quad \text{or} \quad a_2 d \quad (\text{if } d \neq 1).$$

Moreover, if $d \neq 1$ we can take $\sigma = 0$.

IV. Let A be an elementary abelian p-group of order p^2 where p is an odd prime, and let X be the group of automorphisms of A determined by a diagonal but nonscalar subgroup of $\mathrm{GL}(2, p)$. Define G to be the holomorph of A by X.

V. Let P be an extra-special p-group of exponent p, an odd prime, and let n be an integer lying strictly between 1 and p. Let $\{x_\lambda : \lambda \in \Lambda\}$ be a basis for P modulo its centre and define an automorphism t of G by the rule $x_\lambda^t = x_\lambda^n$, $(\lambda \in \Lambda)$. Define G to be the holomorph of P by $\langle t \rangle$.

VI. Let A be a group of type p^∞ with generators $a_1, a_2, \cdots$ and relations $a_{i+1}^p = a_i$ and $a_1^p = 1$. Let Γ be a nonperiodic group of p-adic integers all of which are congruent to 1 modulo p.

(a) Form an extension W of A by Γ using the natural coupling of Γ to A. Let D be a nontrivial elementary abelian p-group of automorphisms of W which

(2) Newman [17] calls these $JN2$-*groups*, however this conflicts with our present terminology.

acts trivially on $\langle a_1\rangle$ and $W/\langle a_1\rangle$. Define G to be the holomorph of W by D. If $p = 2$ and $-1 \in \Gamma$, then in addition require that $t^2 = 1$ or a_1 where $-1 \longrightarrow tA$ in the isomorphism of Γ with W/A. If $[D, t] \neq 1$, the possibility $t^2 = a_1$ can be dispensed with.

(b) Let $p = 2$ and $-1 \in \Gamma$, and choose a group $\langle u\rangle$ of order 2. Form an extension W of $F = A \times \langle u\rangle$ by Γ using the natural coupling of Γ to A, supplemented by $\alpha \longrightarrow (u \longrightarrow u)$ if $\alpha \equiv 1 \bmod 4$ and $\alpha \longrightarrow (u \longrightarrow a_1 u)$ if $\alpha \not\equiv 1 \bmod 4$; moreover, require that $W' = A$ and $t^2 = a_2 u$ where t is as in (a). Finally, form G as in (a) except that D is allowed to be 1 and the centre of G should not contain any element of order 2 except a_1. [This last condition is automatically satisfied in VI(a).]

VII. Let A and Γ be as in VI and let X be an extra-special p-group with generators of order p. Write E for the direct product of A with X in which $\langle a_1\rangle$ and the centre of X are amalgamated.

(a) Form an extension W of E by Γ in which $W' = A$, using the natural coupling of Γ to A supplemented by causing each α in Γ to correspond to an (outer) automorphism of X which acts trivially on $\langle a_1\rangle$ and $X/\langle a_1\rangle$. Let D be an elementary abelian p-group of automorphisms of W which acts trivially on $\langle a_1\rangle$ and $W/\langle a_1\rangle$. Define G to be the holomorph of W by D and suppose D is chosen so that the centre of G contains no elements of order p outside $\langle a_1\rangle$. If $p = 2$ and $-1 \in \Gamma$, require also that $t^2 = 1$ where $-1 \longrightarrow tE$ in the isomorphism of Γ with W/E.

(b) Let $p = 2$ and $-1 \in \Gamma$, and choose a group $\langle u\rangle$ of order 2. Form an extension W of $F = E \times \langle u\rangle$ by Γ using the coupling of Γ to E described in (a), supplemented by $\alpha \longrightarrow (u \longrightarrow u)$ if $\alpha \equiv 1 \bmod 4$ and $\alpha \longrightarrow (u \longrightarrow a_1 u)$ if $\alpha \not\equiv 1 \bmod 4$; moreover require that $W' = A$ and $t^2 = a_2 u$ where $-1 \longrightarrow tF$ in the isomorphism of Γ with W/F. Finally form G as in (a).

VIII. Let X be a soluble T-group and let A be a noncyclic abelian group which is faithful and irreducible as an X-module (so that A contains no proper nontrivial submodules). Define G to be the natural semidirect product of A by X.

IX. Let p be any prime and let F be a subfield of the field of p-adic numbers; denote by Q the field of rational numbers. Choose X to be a group of p-adic integer units in F such that $X \neq \langle -1\rangle$ and $X^+ < Q + X^+ = F$. Define G to be the natural semidirect product of F (as an additive group) by X.

Our principal conclusion is, then,

Theorem 1. *A group is a soluble JNT-group if and only if it is isomorphic with a group of type* I *to* IX.

(1.21) Power automorphisms and soluble T-groups. The proof of Theorem 1 occupies §§3 to 7. Not surprisingly, considerable use will be made of the theory of soluble T-groups. We present next a summary of the relevant facts from this theory.

A fundamental concept is that of a *power automorphism* of a group; this is an automorphism which leaves every subgroup of the group invariant, and so maps each element to a power of itself. A crucial result is that in an abelian group a power automorphism maps elements of the same order to the same power; moreover if an element of infinite order is present in the group, the only nontrivial power automorphism is the involution $a \to a^{-1}$ ([7], [18, §4.1]). An extensive study of power automorphisms of nonabelian groups has been made by Cooper [2].

If G is a soluble T-group, then it is metabelian [18, Theorem 2.3.1]. Also $L = [G', G]$ is the last term of the lower central series, G/L is a *Dedekind group* (i.e. every subgroup is normal) and $C_G(G') = C_G(L)$ is the Fitting subgroup of G.

Noncommutative soluble T-groups are divided into three classes:

(a) periodic groups,

(b) nonperiodic groups of *type* I, i.e. groups in which the centraliser C of the derived subgroup is nonperiodic,

(c) nonperiodic groups of *type* II, i.e. groups in which C is periodic.

If G is a periodic soluble T-group and $L = [G', G]$, then L and G/L do not contain elements with the same odd prime order ([7], [18, Theorem 4.2.2]): also the 2-component of L is radicable.

If G is a soluble T-group of type I and $C = C_G(G')$, then C is abelian and $G = \langle t, C\rangle$ where $|G:C| = 2$, $c^t = c^{-1}$ for all $c \in C$, and $\langle t^2, C^2\rangle = \langle t^2, C^4\rangle$ [18, Theorem 3.1.1].

If G is a soluble T-group of type II, somewhat less is known of its structure. However $C = C_G(G')$ is abelian, G' is radicable and $C = G' \times B$ where B lies in the centre of G. If G' contains an element of prime order p, the p-component of B has finite exponent, say $p^{n(p)}$; if $x \in G$, then x induces in the p-component of C the power automorphism $a \to a^{\alpha}$ where α is a p-adic integer unit satisfying $\alpha \equiv 1 \bmod p^{n(p)}$; here, of course, a^{α} is understood to mean a^{α_1} where α_1 is an integer congruent to α modulo the order of a [18, Theorem 4.3.1].

(1.3) **Remarks on the classification.**

(1.31) **Nilpotent just nonabelian groups.** These groups—which include the extra-special groups of Hall and Higman [10]—occur in our classification and deserve comment.

Let G be a nilpotent just nonabelian group and let Z be its centre. Then, as Newman has shown in [17], G is a p-group, Z is either cyclic or quasi-cyclic and G/Z is elementary abelian; moreover, G' lies in Z and has order p. Let $\{x_\lambda Z: \lambda \in \Lambda\}$ be a basis for G/Z and let $G' = \langle a\rangle$. Then

$$[x_\lambda, x_\mu] = a^{f(\lambda,\mu)}. \tag{1}$$

If we regard G/Z as a vector space over GF(p), then f is a nondegenerate alternating bilinear form. Also $x_\lambda^p \in Z$ and it may be shown that x_λ can be chosen so that either $x_\lambda^p = 1$ or x_λ^p generates Z [17, Lemma 3]. Indeed one can write down generators and relations for G by utilising (1), the position of the x_λ^p and relations sufficient to make $[G', G] = 1$. Newman also proves that if G is countable, it is a central product involving three rather simple kinds of groups [17, Theorem 5].

It should be mentioned that soluble just nonabelian groups which are not nilpotent have also been discussed by Newman [16]; these occur under our type VIII heading unless they are metacyclic.

(1.32) **Faithful irreducible representations.** In connection with groups of type VIII it is desirable to know which soluble T-groups X can act faithfully and irreducibly on an abelian group A. Of course in a situation of this kind A is necessarily isomorphic with the additive group of a vector space, i.e., it is either an elementary abelian p-group or a direct product of copies of the additive group of rational numbers. In the former event the problem reduced to the case where X is abelian. This is because of

Lemma 1. *Let F be a field and let X be a soluble T-group.*

(i) *If X is abelian, it has a faithful irreducible representation over F if and only if there is an extension field E of F such that $X \simeq Y \leq E^*$ and $E = FY$.*

(ii) *In general, X has a faithful irreducible representation over F if and only if the centre of its Fitting subgroup has such a representation.*

Proof. (i) The proof is well known and we omit it.

(ii) We recall first that the Fitting subgroup N of any T-group X (whether soluble or not) is nilpotent and coincides with $C_X(X')$ [18, Lemma 2.2.2]. Write C for the centre of N.

Suppose first that there exists a faithful irreducible (right) FX-module M. Let $0 \neq a \in M$ and $D = C_N(a)$. Since N is nilpotent, D is subnormal in X and hence $D \lhd X$. Therefore D fixes ax for all $x \in X$, which shows that D acts trivially on M and $D = 1$ since M is irreducible and faithful. Consequently M is fixed-point-free with respect to each nonunit element of N.

Assume now that N is nonperiodic. If X is abelian, $C = N = X$ and M is already a faithful irreducible FC-module. Let X be nonabelian—and thus a soluble T-group of type I. It follows from the structure theory of these groups (§1.21) that $N = C$ is abelian and $X = \langle x, C \rangle$ where $a^x = a^{-1}$ if $a \in C$, and $x^2 \in C$. Suppose that M_1 is a proper nonzero FC-submodule of M; then M_1x is also an FC-submodule because $C \lhd X$. Since $x^2 \in C$, we see that $M_1 + M_1x$ and $M_1 \cap M_1x$ are FX-submodules of M; hence $M_1 \cap M_1x = 0$ and $M = M_1 \oplus M_1x$ by the irreduciblility of M. If M_2 is a proper nonzero FC-submodule of M_1, then $M = M_2 \oplus M_2x$ by the same argument, with the result that $M_1 = M_1 \cap (M_2 \oplus M_2x) = M_2$. Therefore M_1

is an irreducible FC-module; it is faithful because M is fixed-point-free.

Now suppose that N is periodic. Let Y be a finitely generated–and therefore finite-subgroup of C. Choose $a \neq 0$ from M; then $(a)FY$ has finite dimension over F, so it contains an irreducible FY-submodule, say L. Since the action of N is fixed-point-free, L is faithful. Consequently, by the first part of this lemma, Y is isomorphic with a finite subgroup of an extension of the field F, which implies that Y is cyclic and C locally cyclic. Let $\bar{F}$ be the algebraic closure of F. Then the torsion-subgroup of $\bar{F}^*$ is a direct product of p^∞-groups, one for each p not equal to the characteristic of F. We can identify C with a subgroup of $\bar{F}^*$. Having done this, define E to be the subfield of $\bar{F}$ generated by C. Since E is algebraic over F, an FC-submodule of E is an ideal. Thus E is an irreducible FC-module, and it is obviously faithful.

Conversely, suppose M_1 is a right FC-module which is faithful and irreducible; the problem is to construct an FX-module that is faithful and irreducible. First we form the induced FX-module

$$I = M_1 \otimes_{FC} (FX)$$

and then we choose an FX-composition series for I; here the term *composition series* (or *system*) is used in the general sense of Kuroš [14, vol. 2, §56]. The composition factors are irreducible FX-modules, so we can assume that none is faithful. Now refine the series to an FC-composition series. If $1 \neq S \lhd X$, then $S \cap C \neq 1$; for $S \cap X' \leq S \cap C$ (since X is metabelian) and $S \cap X' = 1$ implies $S \leq C$. It follows that none of the FC-composition factors can be faithful as FC-modules. It is straightforward to show that an irreducible FC-submodule of I is FC-isomorphic with one of the factors of the FC-composition series, and hence is not faithful.

However, if $\{g_\lambda : \lambda \in \Lambda\}$ is a transversal to C in X, with say $g_{\lambda_0} = 1$, then

$$I \overset{FC}{\simeq} \operatorname*{Dr}_{\lambda \in \Lambda} (M_1 \otimes_{FC} (FC)g_\lambda)$$

and $M_1 \otimes_{FC} (FC)g_{\lambda_0} \overset{FC}{\simeq} M_1$, which gives the contradiction that M_1 is not faithful. This completes the proof.

Returning to the discussion of groups of type VIII, we see that Lemma 1 (with $F = \mathrm{GF}(p)$) gives a complete description of the possible groups X when A is an elementary abelian p-group, although not in group theoretical terms if A is infinite; when A is finite the only restriction on the soluble T-group X is that the centre of $C_X(X')$ be cyclic of order prime to p.

However, when A is a direct product of copies of the additive group of rational numbers (Q), the situation is less clear. Here the following theorem of Baer [1, Proposition] is relevant: *a locally finite group cannot act as an irreducible group of automorphisms of a torsion-free abelian group* $A \neq 1$. This tells us, at least, that X *cannot be periodic.*

(1.33) **Groups of type IX.** While it would probably be difficult to give a purely group theoretic characterisation of G in this case, some possibilities can easily be obtained.

Let F be a subfield of F_p, the field of p-adic numbers, let R_p be the ring of p-adic integers and define $R = F \cap R_p$; now define $X = R^*$. Clearly $X^+ \leq R$. Let $r \in R$; if $r \not\equiv 0 \bmod p$, then $r \in R^* = X$; if $r \equiv 0 \bmod p$, then $1 + r \in R^* = X$ and $r \in X^+$. Therefore $X^+ = R$. Also R is not a field because $1/p \notin R$, so $X^+ \neq F$. Finally, let $f \in F$; we can write $f = q + u$ where $q \in Q$ (the field of rational numbers) and $u \in R_p$. Since $Q \leq F$, we have $u = f - q \in F$, so $u \in F \cap R_p = R$. Consequently, $F = Q + X^+$. The natural semidirect product of F with X is of type IX. For example, we could take $F = F_p$ and $X = R_p^*$.

On the other hand, *in a group of type* IX *the subgroup* X *cannot be periodic.* For suppose that X is periodic; then $\langle -1\rangle \neq X \leq R_p^*$ and the structure of R_p^* show that X is cyclic of order $m > 2$ dividing $p - 1$ where p is odd. Thus $F = Q + X^+$ is a finite extension of Q. If $X = \langle x\rangle$, the irreducible polynomial of x is cyclotomic. Hence each element of F can be written in the form $r + \sum_{i=1}^{n-1} n_i x^i$ where r is rational, n_i integral and $n = \phi(m) > 1$. But x/p has no such representation since $1, x, \cdots, x^{n-1}$ are linearly independent.

2. **Preliminary results.** In this section we shall collect results of a general nature about JNT-groups as well as some technical lemmas necessary for the classification.

(2.1) **Some properties of** JNT**-groups.** Our first result indicates the complex subnormal structure of JNT-groups in general.

Lemma 2. *To each group G there corresponds a JNT-group G^* with subgroups H and K such that $K \lhd H$ and H is subnormal in G^* with subnormal index ≤ 4 while $G \simeq H/K$.*

Proof. The first step is to embed G in a nonunit perfect group G_1 such that G is subnormal in G_1 with subnormal index ≤ 2; that this is possible is a theorem of Dark [3]. Write Z for the set of integers in their natural order and form the standard wreath power $G_2 = \mathrm{Wr}\, G_1^Z$. The order-automorphism $n \to n + 1$ of Z gives rise to an automorphism t of G_2 permuting the copies of G_1 in the same way. Finally, let G^* be the holomorph of G_2 by $\langle t\rangle$.

Let Z_1 and Z_2 be the sets of negative and nonnegative integers in natural order. Then

$$G_2 \simeq (\mathrm{Wr}\, G_1^{Z_1}) \wr (\mathrm{Wr}\, G_1^{Z_2}) \tag{2}$$

where now not all the wreath products are standard; for these and other results about wreath products see P. Hall [9]. The base group L of the wreath product (2) has G_1 as a homomorphic image, so that $G_1 \simeq L/K$ for some $K \lhd L$. Hence there exists a subgroup H of L such that $G \simeq H/K$ and H is subnormal in L with subnormal index ≤ 2. Clearly H is subnormal in G^* with subnormal index ≤ 4.

To show that G^* is a JNT-group observe first that G_2—and hence G^*—is not a T-group. Also it follows by arguments of P. Hall [9, p. 183] that G^* is monolithic(3) with G_2' as its monolith; here G_2 is perfect since G_1 is, so $G_2 = G_2'$. Hence every proper homomorphic image of G^* is abelian.

Corollary. *There exist JNT-groups with unbounded subnormal indices.*

In contrast to this there is

Lemma 3. *A JNT-group G has all its subnormal indices ≤ 2 if one of the following conditions is satisfied.*

(i) *G contains a nontrivial normal abelian subgroup A.*

(ii) *G contains a minimal normal subgroup N which itself contains a minimal normal subgroup N_1.*

Proof. Let H be a nonnormal subnormal subgroup of G. Suppose that (i) is valid. Since $AH \lhd G$, we have

$$[A, H]^g \leq [A, AH] = [A, H]$$

for all $g \in G$. Hence $[A, H] \lhd G$ and $[A, {}_iH] \lhd G$ for each $i \geq 1$. Therefore $H[A, {}_iH] \lhd G$ provided $[A, {}_iH] \neq 1$. Now if s is the subnormal index of H in G, then $[A, {}_sH] \leq H$, so that $[A, {}_sH] = 1$. Let i be the least integer for which $[A, {}_iH] = 1$. Then $i > 0$ and $H[A, {}_{i-1}H] \lhd G$; but also $H \lhd H[A, {}_{i-1}H]$, so the result follows.

Now suppose that (ii) is valid. Clearly N is the direct product of N_1 and certain of its conjugates. Therefore N_1 is simple. If N_1 is abelian, so is N and the result follows from (i). If N_1 is nonabelian, a theorem of Wielandt [22] shows that N normalises H, so $H \lhd HN \lhd G$.

These may be compared with the (more elementary) fact that a non-T-group whose proper subgroups are all T-groups has its subnormal indices ≤ 2 [20].

We shall now consider the possibilities for minimal normal subgroups in JNT-groups.

Lemma 4. *Let G be a JNT-group*:

(i) *the number of minimal normal subgroups of G equals* 0, 1 *or* 2;

(ii) *if this number equals* 2, *both minimal normal subgroups are cyclic of the same prime order*;

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(3) A group is *monolithic* if the intersection of all its nontrivial normal subgroups is nontrivial; this intersection is then called the *monolith*.

(iii) *if G has at least one minimal normal subgroup, each nontrivial normal subgroup of G contains a minimal normal subgroup of G.*

Proof. Throughout H denotes a nonnormal subnormal subgroup of G. Let M be a minimal normal subgroup of G and let N be a nontrivial normal subgroup of G not containing M; thus $M \cap N = 1$. Clearly $M \overset{G}{\simeq} MN/N$ and, since G/N is a T-group, this shows that M is simple. Now $M \leq H$ would imply that $H \lhd G$; therefore $H \cap M = 1$. Also $(H \cap N)M \lhd G$, so that

$$[H \cap N, \ G] \leq ((H \cap N)M) \cap N = H \cap N$$

and $H \cap N \lhd G$. Therefore $H \cap N = 1$. If M is not abelian, a theorem of Wielandt [22] shows that $[H, M] = 1$. In this case

$$H \cap (M \times N) \leq C_{M \times N}(M) = N,$$

and $H \cap (M \times N) = 1$ since $H \cap N = 1$. However this implies that $H = (HM) \cap (HN)$, and since $HM \lhd G$ and $HN \lhd G$, the contradiction $H \lhd G$ is obtained. Thus M is abelian and therefore cyclic of prime order, say p. As we have just seen, $H_1 = H \cap (M \times N)$ cannot be trivial; obviously, H_1 is subnormal in G. Since $H \cap N = 1$,

$$H_1 \simeq H_1N/N \leq (M \times N)/N \simeq M,$$

which shows that H_1 has order p. Next $H \cap M = 1$, so

$$H_1 \simeq H_1M/M \leq (M \times N)/M \simeq N,$$

which shows that N contains a subnormal subgroup N_1 of order p. But $MN_1 \lhd G$, so $[N_1, G] \leq (MN_1) \cap N = N_1$ and $N_1 \lhd G$. Hence N contains a minimal normal subgroup of G, namely N_1. Moreover, if N itself is minimal normal in G, then $N = N_1$ and $|M| = p = |N|$. Thus (ii) and (iii) have been proved.

Finally, let L, M and N be three distinct minimal normal subgroups of G. All three must have the same prime order p. Let $g \in G$; then g induces in the elementary abelian p-group $(M \times N)L/L$ a power automorphism of the form $x \to x^n$. Since $M \overset{G}{\simeq} LM/L$ and $N \overset{G}{\simeq} NL/L$, it follows that g induces $x \to x^n$ in M and in N. Therefore every subgroup of $M \times N$ is normal in G and, in particular, $H \cap (M \times N) \lhd G$. Hence $H \cap (M \times N) = 1$, which is a contradiction. Thus G can have no more than two minimal normal subgroups.

For example, soluble JNT-groups of type IX possess no minimal normal subgroups, those of types I–III and V–VIII have one (and so are monolithic) and those of type IV have two.

A JNT-group with two minimal normal subgroups, and any soluble JNT-group, has its subnormal indices ≤ 2; these statements follow from Lemmas 3 and 4.

Next we record two technical lemmas about JNT-groups which will prove valuable.

Lemma 5. *Let M and N be normal subgroups of a JNT-group G. If one of the following conditions holds, then either M or N is trivial.*

(i) *M and N are periodic and do not contain elements with the same prime order.*

(ii) *M is periodic, N is torsion-free and G/N is periodic.*

Proof. Let H denote a nonnormal subnormal subgroup of G and let $M \neq 1$ and $N \neq 1$. Both (i) and (ii) imply that $M \cap N = 1$; thus $(H \cap M)N \lhd G$ implies that $[H \cap M, G] \leq H \cap M$ and so $H \cap M = 1$. Similarly $H \cap N = 1$. If (i) is valid, $H \cap (M \times N) = (H \cap M) \times (H \cap N) = 1$, which gives $H = (HM) \cap (HN) \lhd G$. If (ii) is valid, then H is periodic since $H \simeq HN/N$; therefore HM is periodic and $(HM) \cap N = 1$, which implies that $(HM) \cap (HN) = H$ and $H \lhd G$.

Lemma 6. *Let $N \lhd G$ where N is abelian and each of its primary components is either elementary abelian or of infinite exponent. Assume that every subgroup of $C_G(N)$ is normal in G. Then G is not a JNT-group.*

Proof. Suppose G is a JNT-group and let H be a nonnormal subnormal subgroup of G. Then $H \nleq C_G(N)$, so there exists $h \in H$ such that $[N, h] \neq 1$. The element h induces a nontrivial power automorphism in N, since $N \leq C_G(N)$. If N is not periodic, $a^h = a^{-1}$ for all $a \in N$ (see §1.21) and $[N, {}_i h] = N^{2^i}$. If s is the subnormal index of H in G, then $H \geq N^{2^s} > 1$, which implies that $H \lhd G$. Thus N is periodic and for some prime p the p-component P is not centralised by h. There is a p-adic integer α such that $a^h = a^\alpha$ for all $a \in P$ [18, Lemma 4.1.2]. With s as before, $H \geq P^{(\alpha-1)^s}$ and consequently $P^{(\alpha-1)^s} = 1$. However this implies that either P is elementary abelian and $\alpha \equiv 1 \bmod p$ or P has infinite exponent and $\alpha = 1$; in each case $[P, h] = 1$.

(2.2) **Splitting criteria.** The following partial generalisation of the Schur-Zassenhaus theorem is well known—see for example [4, Theorem 3] or [18, Lemma 5.1.1]. It will only be required, of course, in situations where G is soluble.

Lemma 7. *Let $N \lhd G$ where G is locally finite and G/N is countable. Suppose that N and G/N do not contain elements with the same prime order. Then N has a complement in G.*

However, we shall encounter situations where this splitting criterion is inadequate. The following result is particularly useful for splitting groups over noncentral minimal normal subgroups; it is based on an idea of M. F. Newman [16].

Lemma 8. *Let A and N be normal subgroups of a group G such that $A \leq N$ and $[A, N] \neq 1$. Assume in addition that N is metabelian and that every subgroup of N/A is normal in G/A. Suppose that A is isomorphic with the additive* *group of a vector space over a prime field and that A contains no proper non-trivial G-invariant subspaces. Then A has a complement in G.*

Proof. We show first that A has a complement in N. Let $C = C_N(A)$; since $[A, N] \neq 1$, we can find a $g \in N \backslash C$. Since every subgroup of N/A is normal in G/A, there is for each $x \in G$ an integer n such that $g^x \equiv g^n \bmod A$. Since A is abelian, $[a, g]^x = [a^x, g^n] \in [A, g]$. It follows that $[A, g] \lhd G$, and for a similar reason $C_A(g) \lhd G$. If A is regarded as a vector space over a prime field, then $[A, g]$ and $C_A(g)$ are subspaces invariant under G. Hence

$$C_A(g) = 1 \quad \text{and} \quad [A, g] = A.$$

The mapping $a \longrightarrow [a, g]$ is therefore an automorphism of A.

Next, define for any $g \in N \backslash C$

$$X_g = \{x\colon x \in N, \ [x, g, g] = 1\}.$$

Let x and y belong to X_g. Since N is metabelian,

$$[xy, g, g] = [[x, g]^y[y, g], g] = [x, g, g^{y^{-1}}]^y = [x, g, g]^y = 1$$

and

$$[x^{-1}, g, g] = [[x, g]^{-x^{-1}}, g] = [x, g, g]^{-x^{-1}} = 1.$$

Thus X_g is a subgroup. Indeed X_g is a complement of A in N. For let $x \in N$ and $a \in A$; then $[ax, g, g] = [[a, g]^x[x, g], g] = [a, g, g]^x[x, g, g]$. Now N/A is a Dedekind group, so it is nilpotent of class ≤ 2 and $[x, g, g] \in A$. Since $a \longrightarrow [a, g]$ is an automorphism of A, it follows that given x in N we can choose a from A so that $ax \in X_g$. Consequently $N = AX_g$. If $x \in A \cap X_g$, then $[x, g, g] = 1$, which implies that $x = 1$. Thus $A \cap X_g = 1$ and X_g is a complement of A in N.

Denote by K any complement of A in N; we shall prove that $K = X_g$ for some $g \in N \backslash C$. Since A is abelian and $N = AK$, there exists a g in K such that $[A, g] \neq 1$. Now $K \simeq N/A$, which is nilpotent of class ≤ 2; hence $K \leq X_g$. Since X_g is also a complement of A, we obtain $K = X_g$.

Next, all complements of A in N are conjugate in N. For consider two such complements X_g and X_h where g and h come from $N \backslash C$. The mapping $a \longrightarrow [g, a, h, h]$ is an automorphism A. Since $[g, h, h] \in A$, there exists an a in A such that

$$[g, h, h]^{-1} = [g, a, h, h]; \tag{3}$$

with this a we compute

$$[g^a, h, h] = [g[g, a], h, h] = [[g, h][g, a, h], h] = [g, h, h][g, a, h, h] = 1$$

by (3). Hence $g^a \in X_h$. Also X_h is nilpotent of class ≤ 2, so $X_h \leq X_{g^a} = (X_g)^a$. Since X_h and X_g^a are both complements of A in N, they are equal.

Finally, we shall show that A has a complement in G. To this end, let K be a complement of A in N and let $g \in G$. Then $K^{g^{-1}}$ is also a complement of A in N, so $K^{g^{-1}} = K^h$ for some $h \in N$. Hence $hg \in N_G(K)$ and

$$G = N(N_G(K)) = (AK)N_G(K) = AN_G(K).$$

But $A \cap N_G(K) = N_A(K) = C_A(K)$, since $A \cap K = 1$. Hence $A \cap N_G(K) = 1$ and $N_G(K)$ is a complement of A in G.

3. **Nilpotent JNT-groups.** Let G be a nilpotent JNT-group; we shall prove that G is of type I. Suppose first that g is an element of infinite order in G such that $\langle g \rangle \lhd G$. Then $\langle g^{2^i} \rangle \lhd G$ and if $i > 2$, the group $G/\langle g^{2^i} \rangle$ is Dedekind and contains elements of order $2^i > 4$. Hence $G/\langle g^{2^i} \rangle$ is abelian and this causes G to be abelian. Consequently the centre of G is periodic and by Lemma 5 it is a p-group for some prime p. Hence there is a minimal normal subgroup N of G which lies in the centre and has order p. Suppose that gN is an element in G/N with infinite order. Then $L = \langle g, N \rangle \lhd G$ and $L = \langle g \rangle \times N$. Thus $L^p = \langle g^p \rangle$ is an infinite cyclic group and $L^p \lhd G$. This is impossible, so G/N—and hence G—is periodic. Therefore G is a p-group.

Assume next that there exists $M \lhd G$ such that $M \neq 1$ and $N \nleq M$. Then $M \cap N = 1$ and at least one of G/M and G/N is hamiltonian, from which it follows that $p = 2$ and G is a 2-group. Let H be a nonnormal subnormal subgroup of G and choose from H an element h of order 2. Then hM and hN generate normal subgroups of G/M and G/N respectively, in each case with order 1 or 2. Therefore h belongs to the centre of G and $\langle h \rangle \lhd G$, which shows that $H \lhd G$. We conclude that G is monolithic with monolith N.

If G/N is abelian, G is just nonabelian and therefore of type I. Assume G/N to be hamiltonian. Then again $p = 2$ and G is a 2-group and $|N| = 2$. Write

$$G/N = (Q/N) \times (E/N)$$

where Q/N is a quaternion group of order 8 and E/N is an elementary abelian 2-group. Let iN, jN and kN be a canonical set of generators for Q/N. Thus $j^i = j^{-1}$ or $j^{-1}a$ where $a \in N$. Since N lies in the centre of G, we obtain $j^{i^2} = j$ in either case. By the same reasoning i^2 commutes with k, with the result that i^2 is in the centre of $Q = \langle i, j, k, N \rangle$. Let $e \in E$; since $[Q, E] \leq N$, the mapping $xN \longrightarrow [x, e]$ is a homomorphism of Q/N into N. The kernel must be nontrivial, so it contains i^2N. Thus $[i^2, E] = 1$ and because $G = QE$ the element i^2 lies in the centre of G. Since $i^2 \neq 1$ and N is the monolith of G, we can conclude that $N < \langle i^2 \rangle$. Moreover iN has order 4, so i has order 8 and $N = \langle i^4 \rangle$. Hence $i^j = i^{-1} = i^7$ or $i^j = i^{-1}i^4 = i^3$; but $i^2 = (i^2)^j = (i^j)^2 = i^6$ in either case, giving the contradiction $i^4 = 1$. This case cannot, therefore, arise.

4. **Soluble JNT-groups without minimal normal subgroups.** Let G be a soluble JNT-group which is not nilpotent and write $L = [G', G]$. Then $L \neq 1$, so G/L is a Dedekind group; L is also the limit of the lower central series of G. We shall assume throughout this section that L *contains no minimal normal subgroups of* G, our aim being to show that G is of type IX. This will be achieved by means of the following programme:

(i) L is torsion-free and abelian.

(ii) L is *rationally irreducible* with respect to G, i.e. every nontrivial normal subgroup of G that is contained in L has periodic factor group in L.

(iii) L is radicable.

(iv) G is of type IX.

If $1 \neq N \lhd G$, then G/N is a soluble T-group and hence is metabelian. Thus $G'' \leq N$ and G'' is either 1 or the monolith of G; the latter is impossible since $G'' \leq L$. Consequently G is metabelian, so G'—and hence L—is abelian.

If L is not torsion-free, there is a prime p such that the subgroup $P = \{a: a \in L, a^p = 1\}$ is not 1. Clearly $P \lhd G$, and since P cannot contain a minimal normal subgroup of G, there is an infinite chain of nontrivial normal subgroups of G,

$$P = P_1 > P_2 > \cdots > P_\alpha > \cdots, \quad (\alpha < \beta),$$

such that

$$\bigcap_{\alpha<\beta} P_\alpha = 1; \tag{4}$$

here β is necessarily a limit ordinal. Let $g \in G$; since G/P_α is a T-group, g induces in P/P_α a power automorphism. Should g centralise P/P_2, it will centralise every P/P_α and hence P; for g must induce in P/P_α an automorphism $a \to a^n$ where n is independent of α, since P is elementary abelian. Therefore $C_G(P/P_2) = C_G(P)$. Consequently $G/C_G(P)$ is cyclic of order dividing $p - 1$, and, if $1 \neq a \in P$, then a^G is finitely generated and therefore finite. However this would imply that a^G contained a minimal normal subgroup of G, contrary to hypothesis.

Next we prove that L is rationally irreducible with respect to G. If this is false, there surely exists an infinite chain of nontrivial normal subgroups of G,

$$L = L_1 > L_2 > \cdots > L_\alpha > \cdots, \quad (\alpha < \beta),$$

such that L/L_2 is not periodic and

$$\bigcap_{\alpha<\beta} L_\alpha = 1; \tag{5}$$

again β is a limit ordinal. Let $\alpha \geq 2$; now $(G/L_\alpha)' = G'/L_\alpha \geq L/L_\alpha$, so G/L_α is a soluble T-group of type I. Let $C = C_G(L)$; then certainly C centralises L/L_α. Now if H is any T-group, $C_H(H') = C_H([H', H])$ [18, Lemma 2.2.2]. From this we deduce that $C \leq C_G(G'/L_\alpha)$. Let $g \in G \setminus C$—note that $C \neq G$ because G is not nilpotent. Then g does not centralise G'/L_α if α is large enough. From the structure of soluble T-groups of type I (§1.21) it follows that C/L_α is abelian and g induces in C/L_α the automorphism $a \to a^{-1}$ for each $\alpha \geq 2$. By (5), C is abelian and $c^g = c^{-1}$ for all $c \in C$ and $g \in G \backslash C$. This implies that every subgroup of C is normal in G and Lemma 6 yields the contradiction that G is not a JNT-group.

We wish now to establish that L is radicable. Supposing this to be false, we can find a prime p such that $L^p < L$. Here p must be odd, for L/L^2 is radicable by Lemma 2.4.1 of [18]. Now G/L^p is a nonabelian soluble T-group in which the elements of finite order form a subgroup (since L/L^p is periodic and G/L nilpotent). If G/L is not periodic, G/L^p is soluble T of type II [18, Corollary 2, Theorem 3.1.1] and L/L^p is radicable [18, Theorem 4.3.1]. This is impossible, so G/L is periodic.

If L^{p^ω} is the intersection of all the subgroups L^{p^i}, then L/L^{p^ω} is torsion-free since L is; the rational irreduciblility of L now implies that $L^{p^\omega} = 1$. If $x \in D = C_G(L/L^p)$, then x induces in each L/L^{p^i} an automorphism with order a power of p. Since G/L is periodic, we conclude that x induces in L an automorphism with order a power of p. If $C = C_G(L)$, then D/C is a p-group. However G/L^p is a periodic soluble T-group and therefore G/L can have no elements of order p [18, Theorem 4.2.2]. Since $L \leq C$, it follows that $C = D$. Therefore G/C is a cyclic group of order dividing $p-1$. Let $1 \neq a \in L$ and set $A = a^G$. Then A is free abelian of finite rank. Suppose now that G/L contains an element with odd prime order q. Then G/A^q is periodic and G/L and L/A^q both contain an element of order q, contradicting Theorem 4.2.2 of [18]. Consequently G/L is a 2-group.

Let g be any element of $G \backslash C$. Then g induces in A/A^{3^i} a power automorphism whose order is a power of 2 and divides $\phi(3^i) = 2 \cdot 3^{i-1}$; hence g must induce the identity or $a \to a^{-1}$, the latter being the only power automorphism of order 2. The intersection of all the A^{3^i} is 1 since A is free abelian; thus $a_1^g = a_1^{-1}$ for all a_1 in A unless $[A, g] = 1$. Since L/A is periodic and L is torsion-free, it follows that

$$a^g = a^{-1} \quad (a \in L,\ g \in G \backslash C). \tag{6}$$

Next $L = L^2$; thus L/A^2 has a subgroup of type 2^∞. Let i be an integer > 2 and let P/A^{2^i} denote the subgroup of all elements of L/A^{2^i} which have odd order. Then $P < L$ and L/P is a radicable abelian 2-group. C/A^{2^i} centralises L/A^{2^i}, and hence G'/A^{2^i}, so it is Dedekind; but C/A^{2^i} also has a factor of type 2^∞, which causes it to be abelian. Therefore C is abelian. Let us write

$$C/A^{2^i} = (P/A^{2^i}) \times (E/A^{2^i}) \tag{7}$$

where E/A^{2^i} is a 2-group—recall here that G/L is a 2-group. G/P is a 2-group which is not Dedekind. Let $g \in G \setminus C$; then g cannot centralise G'/P since elements of L/P are transformed by g into their inverses and L/P is radicable. From the structure of soluble 2-groups with T [18, Lemma 4.2.1] we know that g^2P belongs to the centre of G/P and hence g^2 centralises E/A^{2^i}. Also g^2 centralises L by (6), so g^2 centralises P. By (7), C/A^{2^i} is centralised by g^2 for each i; thus g^2 centralises C. In addition, g induces a nontrivial power automorphism in C/P since g does not centralise L/P. Once again we invoke the structure of soluble 2-groups with T and conclude that every g in $G \setminus C$ induces $a \longrightarrow a^{-1}$ in C/P. If $c \in C$ and $g \in G \setminus C$, then $c^g = c^{-1}a$ where $a \in P$; therefore

$$c = c^{g^2} = (c^g)^{-1}a^{-1} = (c^{-1}a)^{-1}a^{-1} = ca^{-2}$$

since C is abelian. Since L is torsion-free, $a = 1$ and $c^g = c^{-1}$. It follows that every subgroup of C is normal in G. Lemma 6 provides the contradiction that G is not a JNT-group. L is therefore radicable.

Lemma 8 can now be applied with $A = L$ and $N = G$; observe here that by rational irreducibility there are no proper nontrivial radicable subgroups of L that are normal in G. Hence L has a complement in G, say X;

$$G = LX \quad \text{and} \quad L \cap X = 1.$$

Suppose that X does not act faithfully on L, i.e. $D = C_X(L) \neq 1$. Notice that $D \lhd LX = G$ and that G/D is a T-group. Now $G' \nleq D$ and $G'D/D \geq LD/D \simeq L$, so G/D is a soluble T-group of type I. If $g \in G \setminus C$, then $a^g = a^{-1}$ for all a in L in view of the structure of soluble T-groups of type I and because L and LD/D are isomorphic as G-operator groups. Rational irreducibility now forces L to be isomorphic with Q, the additive group of rational numbers. Let M be a subgroup of L such that L/M is isomorphic with Q/Z, where Z is the subgroup of all integers. Then $M \lhd G$ and C/M is abelian. An element g of $G \setminus C$ induces $a \longrightarrow a^{-1}$ in L and thus in L/M; also g induces a power automorphism in C/M which must agree with $a \longrightarrow a^{-1}$ on L/M. Since L/M has elements of every finite order and power automorphisms map elements of the same order to the same power, g must induce $a \longrightarrow a^{-1}$ also in C/M. The intersection of all subgroups like M is 1, which shows that C is abelian and $a^g = a^{-1}$ for all $a \in C$. Hence every subgroup of C is normal in G. Lemma 6 now gives a contradiction, so X acts faithfully on L. Since the group G is metabelian, $[L, X'] = 1$; therefore X is abelian and $G' = L$.

Let Q denote the field of rational numbers. Choose $a \neq 1$ from L; then L is an irreducible QX-module and $L = a^{QX}$. The mapping $r \longrightarrow a^r$ is a homomorphism of QX-modules from QX onto L with kernel K, a maximal ideal of QX;

thus QX/K is a field. Let $A = a^X$; then $A \lhd G$ and G/A is a T-group. If $x \in X$ and n is a nonzero integer, then

$$(a^{1/n}A)^x = a^{m/n}A$$

for some integer m. Therefore $(1/n)x - m/n \in ZX + K$, where ZX is the integral group ring of X, and

$$QX = Q + ZX + K. \tag{8}$$

Since L is not a minimal normal subgroup of G, there is a prime p and a normal subgroup P of G such that P is properly contained in L and L/P is a p-group. Let I be the intersection of all such P. If $I \neq 1$, then L/I is a p-group since it is periodic. Hence $I = I^q$ for all primes $q \neq p$; but $I = I^p$ by minimality of I, so in fact I is radicable. Thus $I = 1$.

Next let L/P_1 and L/P_2 be nontrivial p-groups where $P_1 \lhd G$ and $P_2 \lhd G$. An element x of X induces in L/P_1 and L/P_2 power automorphisms that can be described by p-adic integers α_1 and α_2. But $L/P_1 \cap P_2$ is also a p-group and x induces in it a power automorphism describable by a p-adic integer α_3. Clearly $\alpha_1 = \alpha_3$ and $\alpha_2 = \alpha_3$, so $\alpha_1 = \alpha_2$. It follows that to each x in X there corresponds a unique p-adic integer unit α_x such that $b^x P = (bP)^{\alpha_x}$ for all b in L and all $P \lhd G$ with $P \leq L$ and L/P a p-group. Moreover, $\alpha_x = 1$ if and only if $x = 1$ since $I = 1 = C_X(L)$.

This enables us to construct a mapping α from QX to F_p, the field of p-adic numbers, as follows:

$$\left(\sum_{x \in X} r_x x\right)\alpha = \sum_{x \in X} r_x \alpha_x \qquad (r_x \in Q).$$

α is a ring homomorphism because $\alpha_{xy} = \alpha_x \alpha_y$; also it is easy to verify that Ker $\alpha = K$, using the fact that $I = 1$. Let F be the image of QX under α; then

$$QX/K \simeq F \leq F_p,$$

the isomorphism being of rings. Therefore F is a subfield of F_p. Define Y to be the image of X under α; then Y consists of p-adic integer units. Let $\overline{G}$ be the semidirect product of F (*qua* additive group) by Y. Then the mapping $a^r x \to ((r)\alpha, \alpha_x)$ is easily seen to be an isomorphism of G with $\overline{G}$.

Finally, $\overline{G}$ is of type IX. For $(ZX)\alpha = Y^+$, the additive subgroup generated by Y, and by (8), $F = Q + Y^+$; on the other hand, $F = Y^+$ would imply that L is minimal normal in G. Also $Y \neq \langle -1 \rangle$ since otherwise $F = Q$ and G would be a T-group.

5. **Nonperiodic soluble JNT-groups which contain a minimal normal subgroup.** Throughout this section G will denote a nonnilpotent, nonperiodic, soluble JNT-group such that $L = [G', G]$ contains a minimal normal subgroup of G, say N.

Of course, N is either an elementary abelian p-group or a direct product of copies of the additive group of rational numbers.

(5.1) **Case** N **torsion-free.** Observe first that G is monolithic with monolith N. For otherwise there is a normal subgroup $M \neq 1$ such that $M \cap N = 1$. Since $N \overset{G}{\simeq} MN/M$, the subgroup MN/M is minimal normal in the soluble T-group G/M and hence is simple; therefore N is cyclic of prime order, contrary to assumption.

Suppose that G splits over N and that X is a complement of N. Then $C_X(N) \lhd G$ and, G being monolithic, this implies that $C_X(N) = 1$ and X acts faithfully on N. Finally X is a soluble T-group and N is not cyclic, so G is of type VIII.

Consider next what happens if G does not split over N. In this situation Lemma 8 shows that

$$[N, G'] = 1. \tag{9}$$

G/N cannot be abelian; for if it were, G' would equal N and, since $L = [G', G] \neq 1$, Lemma 8 would imply that G splits over N. Also, $G/C_G(N)$ is essentially an irreducible group of automorphisms of N; since N is torsion-free, a theorem of Baer [1, Proposition] shows that $G/C_G(N)$—and hence G/N—cannot be periodic. If G/N were a soluble T-group of type I, then G/G' would be periodic [18, 3.1] and, by (9) so would $G/C_G(N)$. It follows that G/N is a soluble T-group of type II and G'/N is periodic. Commutation with a fixed element of G' gives rise to a homomorphism of G'/N into N since $[N, G'] = 1$. However $\mathrm{Hom}(G'/N, N) = 0$ because G'/N is periodic and N is torsion-free. Consequently G' is abelian. Now, N being radicable, we can write $G' = N \times R$ where $R \simeq G'/N$. Clearly R is the subgroup of all elements in G' with finite order and $R \lhd G$. But N is the monolith of G, so $R = 1$ and $G' = N$, contradicting the noncommutativity of G/N. Hence this case cannot arise.

(5.2) **Case** N **an elementary abelian** p**-group and** $[N, G'] \neq 1$. Here Lemma 8 can be applied directly to show that G splits over N; let X be a complement of N and suppose that $C = C_X(N) > 1$. Since $C \lhd G$, we can assert that G/C is a soluble T-group. Moreover $N \overset{G}{\simeq} NC/C$, so N is cyclic of order p. It follows that G' centralises N. This is impossible, so $C = 1$ and G is of type VIII—note that N is not cyclic.

(5.3) **Case** N **an elementary abelian** p**-group and** $[N, G'] = 1$. This case leads to several different types of groups and will accordingly be analysed under several subheadings. Since N, but not G, is periodic, G/N is abelian or soluble T of type I or soluble T of type II.

(5.31) **Case** G/N **abelian.** Here $G' = N$ and G splits over N by Lemma 8. Let X be a complement of N in G and suppose that $C = C_X(N) \neq 1$. Then NC/C—and hence N—is cyclic of order p. Therefore X/C is finite and C must be nonperiodic. Let F be a maximal torsion-free subgroup of C; then $F \lhd G$ since $[N, F] = 1$ and X is abelian. Also, C/F is periodic by maximality of F; consequently G/F is periodic. However, Lemma 5 yields the contradiction $N = 1$ or $F = 1$. It follows that $C = 1$. If N were cyclic, X would be cyclic with order dividing $p - 1$ and G would be finite. Hence N is not cyclic and G is of type VIII.

(5.32) **Case G/N a soluble T-group of type I.** From the structure of soluble T-groups of type I it is seen that G/G' is periodic and G'/N nonperiodic and abelian. Choose a maximal torsion-free subgroup F/N of G'/N; then $F \lhd G$ because G/N is a T-group; also G/F is periodic. By hypothesis $[N, G'] = 1$; thus $F' \leq N \leq \zeta(F)$. Therefore, for any x, y in F, $1 = [x, y]^p = [x^p, y]$ and $F^p \leq \zeta(F)$; in particular, F^p is abelian. Since F/N is torsion-free and N elementary abelian, $H = (F^p)^p$ is torsion-free. But clearly $H \lhd G$ and G/H is periodic. Lemma 5 once again gives a contradiction, showing that this case cannot arise.

(5.33) **Case G/N a soluble T-group of type II.** Here G'/N is a nontrivial, periodic radicable group; moreover, if $C = C_G(G'/N)$, then C/N is periodic and abelian. Two possibilities must now be distinguished.

(5.331) **Case $[N, C] \neq 1$.** By Lemma 8 the group G splits over N; let X be a complement of N and assume that $D = C_X(N) \neq 1$. Then G/D is a soluble T-group, so it is metabelian; also $N \cap D = 1$, showing that G is metabelian. Next $(G')^p \neq 1$ since G'/N is radicable; therefore $G/(G')^p$ is a soluble T-group in which the elements of finite order form a proper subgroup; such a group cannot be of type I. Consequently, either $G' = (G')^p$ or $G/(G')^p$ is soluble T of type II, in which event $G'/(G')^p$ is radicable, an obvious absurdity. It follows that $G' = (G')^p$. Also G' is periodic and abelian; with the aid of Lemma 5, we deduce that G' is a radicable abelian p-group. If $c \in C$, the mapping $a \to [a, c]$ is a homomorphism of G' into N since $[G', C] \leq N$ and $[N, G'] = 1$. But $\mathrm{Hom}(G', N) = 0$ because G' is radicable and N elementary abelian. Hence $[G', C] = 1$ and $[N, C] = 1$, which is contrary to hypothesis. Therefore $D = 1$ and G is again of type VIII.

(5.332) **Case $[N, C] = 1$.** We shall show that G is of type VI or VII by pursuing the following programme:

(i) G' is a radicable abelian p-group.

(ii) G is monolithic with monolith N.

(iii) Properties of $C = C_G(G'/N)$.

(iv) C is abelian and G is of type VI.

(v) C is nilpotent of class 2 and G is of type VII.

Since C/N is abelian, C is nilpotent of class ≤ 2. Lemma 5 implies that C is also a p-group. Commutation with a fixed element of C produces a homomorphism of G'/N into N since $[G', C] \leq N$ and $[N, C] = 1$. But $\mathrm{Hom}(G'/N, N) = 0$, so that

$$(10) \qquad [G', C] = 1 \quad \text{and} \quad C = C_G(G').$$

In particular G' is abelian. Furthermore $G' = (G')^p$ since otherwise $G/(G')^p$ would be soluble T of type II. Therefore G' is a radicable abelian p-group.

Let $g \in G$ and denote by τ_g the automorphism induced by g in G'. Now g induces in G'/N a power automorphism which can be described by a p-adic integer unit, say α_g. Writing θ for the power automorphism $a \longrightarrow a^{\alpha_g}$ of G', we see that $\tau_g^{-1}\theta$ acts trivially on G'/N. Therefore $\tau_g^{-1}\theta - 1 \in \operatorname{Hom}(G', N) = 0$ and $\tau_g = \theta$. Consequently

$$a^g = a^{\alpha_g} \qquad (a \in G', g \in G). \tag{11}$$

The next point to establish is that G is monolithic with monolith N; since (11) shows every subgroup of G' to be normal in G, it will then follow that G' is a p^∞-group. Suppose there exists $M \lhd G$ such that $1 \neq M$ and $N \cap M = 1$. If $C \cap M = 1$, then $[G', M] \leq C \cap M = 1$ and $M \leq C$. This cannot be, so $C \cap M \neq 1$ and there is no loss in assuming that $M \leq C$. Now G/C—and hence G/M—is nonperiodic and G/M is not abelian since $N \leq G'$. Hence G/M is also a soluble T-group of type II. It follows that C/M—and hence C—is abelian. Let $g \in G$; then g induces in G'/N—*and therefore in* C/N—the automorphism $a \longrightarrow a^{\alpha_g}$; here we use the radicability of G'/N. But g also induces $a \longrightarrow a^{\alpha_g}$ in G' by (11), and therefore in $G'M/M$ and C/M. Hence $a^g = a^{\alpha_g}$ for all $a \in C$. Since $C = C_G(G')$ and G' has infinite exponent, Lemma 6 provides a contradiction. Thus we conclude that G is monolithic with monolith N.

Let Z be the centre of C; then $G' \leq Z$ by (10) and since G' is radicable,

$$Z = G' \times D_1 \tag{12}$$

for some $D_1 \leq Z$. Clearly $D_1N \lhd C \lhd G$, so $D_1N \lhd G$ and $[D_1, G] \leq G' \cap (D_1N)$, which shows that

$$[D_1, G] \leq N. \tag{13}$$

If $d \in D_1$ and $g \in G$, then $1 = [d, g]^p = [d^p, g]$; therefore $D_1^p \leq \zeta(G)$ and $D_1^p \lhd G$; since N is the monolith of G, it follows that $D_1^p = 1$ and D_1 is an elementary abelian p-group.

Write T/G' for the torsion-subgroup of G/G'; then $C \leq T$ since C/N is periodic. Moreover $C \neq G'$ by Lemma 6. The structure of soluble T-groups of type II provides the following information: C/G' has finite exponent p^e and $\alpha_g \equiv 1 \bmod p^e$ for all $g \in G$ [18, Theorem 4.3.1]. Observe that $e > 0$, so that

$$\alpha_g \equiv 1 \bmod p. \tag{14}$$

Now suppose that $g \in T$; since α_g is a p-adic integer unit of finite order satisfying (14), there are the following possibilities: either p is odd and $\alpha_g = 1$ or $p = 2$ and $\alpha_g = \pm 1$. Thus T/C has order 1 or 2 and we can write

$$T = \langle t, C \rangle \tag{15}$$

where either $t = 1$ or $\alpha_t = -1$; in either case $t^2 \in C$ and t^2N has order 1 or 2; therefore t^2 has order dividing 4. Next we show that t^2 *is in the centre of* G; let $t \neq 1$. If $g \in G$ then $t^g = ta$ where $a \in G'$. Hence $(t^2)^g = (ta)^2 = t^2$ since $a^t = a^{-1}$. In particular $\langle t^2 \rangle \lhd G$. If $t^2 \neq 1$, then $N \leq \langle t^2 \rangle$. Let $p = 2$ and $G' = \langle a_1, a_2, \cdots \rangle$ where $a_{i+1}^2 = a_i$ and $a_1^2 = 1$; then $N = \langle a_1 \rangle$. Clearly $t^2 \in Z$; let us consider the position of t^2 in Z. If $t^2 \in G'$, then $t^2 = 1$ or a_1 since $a_2^t = a_2^{-1}$ if $t \neq 1$. Suppose $t^2 \notin G'$; then since t^2 has order 2 or 4, it belongs to $\langle a_2 \rangle \times D_1$, and since $a_1 \in \langle t^2 \rangle$, we can assume that $t^2 = a_2 u$ where $1 \neq u \in D_1$. Thus the possibilities for t^2 are 1, a_1 and $a_2 u$.

The group T/G' has finite exponent and this is well known to imply that T/G' is a direct factor of G/G' (see [13, Theorem 8]); let

$$G/G' = (T/G') \times (Y/G'). \tag{16}$$

From (16) and (15) we obtain

$$G = \langle t, C, Y \rangle. \tag{17}$$

Consider the case when C *is abelian*. Here $C = Z = G' \times D_1$ and (17) becomes $G = \langle t, D_1, Y \rangle$. Now set $W = \langle t, Y \rangle$. Since D_1 is elementary abelian, we can write $D_1 = (\langle t, G' \rangle \cap D_1) \times D$, say. Hence $G = \langle t, D, Y \rangle = WD$—observe that $W \lhd G$ since $G' \leq Y \leq W$. Since $T \cap Y = G'$, we have $W \cap D \leq \langle t, G' \rangle \cap D = 1$. Hence $G = WD$ and $W \cap D = 1$.

Next we analyse the structure of W. First

$$W/W \cap C \simeq WC/C = G/C. \tag{18}$$

The map $gC \longrightarrow \alpha_g$ is an isomorphism of G/C with a nonperiodic group Γ of p-adic integers all of which are congruent to 1 modulo p. Now $W \cap C = W \cap (G' \times D_1) = G' \times (W \cap D_1)$. Also

$$W \cap D_1 = \langle t, Y \rangle \cap D_1 = \langle t, G' \rangle \cap D_1 = \langle t^2, G' \rangle \cap D_1.$$

If $t^2 \in G'$, then $W \cap D_1 = 1$ by the last equation; otherwise $t^2 = a_2 u$ and $\langle t^2, G' \rangle \cap D_1 = \langle u \rangle$, so $W \cap D_1 = \langle u \rangle$. Hence $W \cap C = G'$ or $G' \times \langle u \rangle$ according as $t^2 \in G'$ or $t^2 \notin G'$. Also, $W/W \cap C \simeq \Gamma$ by (18). Suppose that $t^2 = a_1$; if there exists a d in D such that $t^d \neq t$, then $d^t = da_1$ by (13); thus $(td)^2 = t^2 a_1 = 1$, and, replacing t by td, we can assume that $t^2 = 1$. In other words, we can exclude $t^2 = a_1$ unless $[D, t] = 1$. Suppose $t^2 \notin G'$; then $p = 2$ and $t^2 = a_2 u, (u \in D_1)$. Let $g \in G$; since t^2 belongs to the centre of G, we have $(a_2 u)^g = a_2 u$ and $u^g = a_2^{1-\alpha_g} u$. Thus $u^g = u$ or $a_1 u$ according as $\alpha_g \equiv 1 \bmod 4$ or $\alpha_g \not\equiv 1 \bmod 4$.

Finally, D centralises N and W/N. If $t^2 \in G'$, then in addition $D \neq 1$; for $W' = G'$ and $C_W(W') = W \cap C = G'$; thus W is not a JNT-group by Lemma 6 and $D \neq 1$. The centre of G contains no element of order p except a_1 since N is the monolith. G is of type VI(a) or (b) according as $t^2 \in G'$ or $t^2 \notin G'$.

The remaining possibility under heading (5.332) is that C *is nilpotent of class exactly* 2. The centre of C is Z and C/N is abelian; thus

$$N = C' < G' \leq Z < C.$$

If $x, y \in C$, then $1 = [x, y]^p = [x^p, y]$, showing that $C^p \leq Z$ and C/Z is an elementary abelian p-group.

Let us prove next that $C^p = G'$. Now $G' \leq C^p$ is immediate, and since C^p is abelian, we can write $C^p = G' \times F$. Suppose x is a nontrivial element of F; then $x = c^p a$ for some $c \in C$ and $a \in N$; for C/N is abelian. Let g be any element of G; then $c^g = c^{\alpha_g} b$ for some $b \in N$. Hence

$$(c^p)^g = (c^{\alpha_g} b)^p = (c^p)^{\alpha_g}$$

and $\langle c^p \rangle \lhd G$. Now $a \neq x$ since $G' \cap F = 1$; thus $c^p \neq 1$ and $a \in \langle c^p \rangle$. Therefore $x = c^p a \in \langle c^p \rangle$ and $\langle x \rangle \lhd G$. This gives the contradiction $N \leq \langle x \rangle$. Hence $F = 1$ and $C^p = G'$.

Let $\{x_\lambda Z \colon \lambda \in \Lambda\}$ be a basis for the elementary abelian p-group C/Z and set $X = \langle x_\lambda \colon \lambda \in \Lambda \rangle$. Then $C = XZ = XG'D_1$. Let $y \in (XG') \cap D_1$ and write $y = x_{\lambda_1}^{n_1} \cdots x_{\lambda_r}^{n_r} a$ where $a \in G'$, the n_i are integers and the λ_i are distinct elements of Λ. Then $x_{\lambda_1}^{n_1} \cdots x_{\lambda_r}^{n_r} \in Z$ and the linear independence of the $x_{\lambda_i} Z$ implies that $p | n_i$ for $i = 1, \cdots, r$. Hence $y \in C^p G' = G'$ and $y \in G' \cap D_1 = 1$. Writing $E = XG'$ we obtain $C = E \times D_1$.

Suppose that $x_\lambda^p \neq 1$. Since $G' = C^p$, there is an element a of G' such that $x_\lambda^p = a^p$. This implies that $(x_\lambda a^{-1})^p = 1$. Replacing x_λ by $x_\lambda a^{-1}$ we may assume that $x_\lambda^p = 1$ for all $\lambda \in \Lambda$. This implies that $X \cap Z = N$: for let $y \in X \cap Z$; since $X' \leq N$, we can write $y \equiv x_{\lambda_1}^{n_1} \cdots x_{\lambda_r}^{n_r} \bmod N$ where the λ_i are distinct. Thus $p | n_i$ for all i and $y \in N$ as required. Next N is actually the centre of X since $\zeta(X) \leq X \cap Z = N$. Thus $X' = \zeta(X) = N$ and $X/N \simeq C/Z$. Consequently X is an extraspecial p-group. Since $X \cap G' = N = \langle a_1 \rangle$, the group E is a direct product of X and G' in which the centre of X and $\langle a_1 \rangle$ are amalgamated.

Consider now the position of t^2; if $t^2 \notin G'$, then $p = 2$ and $t^2 = a_2 u$ where $u \in D_1$. If $t^2 \in G'$ and $t^2 \neq 1$, then $p = 2$ and $t^2 = a_1$. Suppose that $[X, t] \neq 1$; then $[x_\lambda, t] \neq 1$ for some $\lambda \in \Lambda$. Since $x_\lambda N$ has order 2, it is centralised by t and $x_\lambda^t = x_\lambda a_1$; thus $(t x_\lambda)^2 = a_1^2 x_\lambda^2 = 1$. If, however, $[X, t] = 1$, write $a_1 = x^2$ for some $x \in X$ (note $N = X^2$); then $(tx)^2 = 1$. Therefore, if $t^2 \in G'$, we may assume that $t^2 = 1$.

From $C = XZ$ and equations (12) and (17) we obtain $G = \langle t, X, Y, D_1 \rangle$. Now define $W = \langle t, X, Y \rangle$. Writing

$$D_1 = (\langle t, G' \rangle \cap D_1) \times D,$$

we have $G = WD$ and $W \lhd G$. Suppose that $w \in W \cap D$ and, using $G' \leq Y$, write

$w = t^i xy$ where $x \in X$ and $y \in Y$; then $y \in T \cap Y = G'$ and $t^i \in C$, which shows that i may be assumed even. Since $t^2 \in Z$, it follows that $x \in X \cap Z = N$ and $w \in \langle t, G' \rangle \cap D = 1$. Hence $W \cap D = 1$.

We turn now to the structure of W. First $W \cap C = W \cap (E \times D_1) = E \times (W \cap D_1)$; moreover, as above,

$$W \cap D_1 = \langle t, X, G' \rangle \cap D_1 = \langle t^2, X, G' \rangle \cap D_1 = \langle t^2, G' \rangle \cap D_1.$$

If $t^2 \in G'$, then $W \cap D_1 = 1$; otherwise $t^2 = a_2 u$ and $W \cap D_1 = \langle u \rangle$. Thus $W \cap C = E$ or $E \times \langle u \rangle$ according as $t^2 \in G'$ or $t^2 \notin G'$. Moreover $W/W \cap C \simeq G/C$ and G/C is isomorphic with Γ, a group of p-adic integers all of which are congruent to 1 modulo p. If $t^2 \notin G'$ then one shows (as in the case C abelian) that $u^g = u$ or $a_1 u$ according as $\alpha_g \equiv 1 \bmod 4$ or $\alpha_g \not\equiv 1 \bmod 4$. Since $\alpha_g \equiv 1 \bmod p$ for all g in G, the factor X/N is central in G and $[X, G] \leq N$.

Finally D acts faithfully on W and centralises W/N and N. Therefore G is of type VII(a) or (b) according to whether t^2 is or is not in G'.

6. **Periodic soluble JNT-groups.** Throughout this section G will denote a non-nilpotent JNT-group which is both soluble and periodic; N is a minimal normal subgroup of G contained in $L = [G', G]$. Thus N is an elementary abelian p-group for some prime p.

(6.1) **Case** $[N, G'] \neq 1$. Here G is of type VIII; the argument is that of (5.2).

(6.2) **Case** $[N, G'] = 1$. Since G'/N is abelian, G' is nilpotent in this case; hence G' is a p-group by Lemma 5. Consequently G has a unique Sylow p-subgroup P containing G'.

We shall require the equation

$$[N, C_P(G'/N)] = 1. \tag{19}$$

To prove this let x in P centralise G'/N. Since $C_G(G'/N)/N$ is nilpotent, $\langle x, N \rangle \lhd G$, which shows that $[N, x] \lhd G$. Assuming that $[N, x] \neq 1$, we find that $N = [N, x]$ since N is minimal normal in G. If p^r is the order of x, then $N = [N, {}_{p^r}x] = [N, x^{p^r}] = 1$, a contradiction.

(6.21) **Case** $[N, G'] = 1$ **and** P/N **abelian.** Here P centralises G'/N since $G' \leq P$; therefore

$$[N, P] = 1 \tag{20}$$

by (19). Also $P' \leq N$ and (20) shows that P is nilpotent of class at most 2. Since G/P is abelian, $[(C_G(P))', C_G(P)] = 1$ and $C_G(P)$ is nilpotent. Lemma 5 shows that $C_G(P)$ is a p-group; hence

$$C_G(P) \leq P. \tag{21}$$

Now it is necessary to distinguish two subcases.

(6.211) **Case** P **abelian.** Suppose that P contains an element of order p^2

and define P_1 to be the subgroup of all a in P such that $a^{p^2} = 1$. Thus $1 \neq P_1^p \lhd G$ and G/P_1^p is a T-group. Let $g \in G$; then g induces a power automorphism $a \to a^\alpha$ in P/P_1^p—here α is a p-adic integer unit. Writing ξ for the automorphism of P induced by g and η for the power automorphism $a \to a^\alpha$ of P, we see that $\theta = \xi^{-1}\eta$ is an automorphism of P which acts trivially on P/P_1^p. Therefore, for any $a \in P$ we have $a^\theta = ab^p$ for some $b \in P_1$; hence $(a^p)^\theta = (ab^p)^p = a^p$. Therefore $(b^p)^\theta = b^p$; from this and $a^\theta = ab^p$ it follows that $a^{\theta^p} = ab^{p^2} = a$. Hence $1 = \theta^p = \xi^{-p}\eta^p$ since a power automorphism commutes with every automorphism of P. Now by (21) we have $C_G(P) = P$, so the order of ξ is prime to p. Consequently $\xi^p = \eta^p$ implies that $\xi \in \langle\eta\rangle$ and ξ is a power automorphism of P. By Lemma 5.2.2 of [18] the group G is a T-group. In view of this contradiction P is an elementary abelian p-group.

Next it will be shown that G splits over P. By hypothesis $C_G(N) \geq G'$; therefore $G/C_G(N)$ is abelian. $G/C_G(N)$ can be regarded as an irreducible group of linear transformations of N *qua* vector space. This implies that $G/C_G(N)$ is isomorphic with a periodic subgroup of the multiplicative group of a field. Therefore $G/C_G(N)$ is locally cyclic and, in particular, countable (see [6, p. 296]). Since elements of G induce power automorphisms in the elementary abelian p-group P/N, the group $C_G(N)/(C_G(N) \cap C_G(P/N))$ is cyclic of order dividing $p - 1$. If g centralises both N and P/N, then g induces in P an automorphism of order 1 or p. But $P = C_G(P)$, so $g \in P$. Thus

$$C_G(N) \cap C_G(P/N) = C_G(P) = P \tag{22}$$

and G/P is countable. Lemma 7 shows that G splits over P, say $G = PX$ and $P \cap X = 1$. Moreover X acts faithfully on P in view of (22).

Consider the situation when N is the monolith of G. Assume that $N \neq P$ and let $a \in P\backslash N$. If a^G were finite, it would be a direct product of minimal normal subgroups of G by Maschke's theorem—observe that X has no elements of order p. This is consistent only with $a^G = N$ because N is the monolith. Therefore a^G must be infinite—and hence so is X. Since $X/C_X(P/N)$ is finite, $C_X(P/N) \neq 1$. Also $C_X(P/N) \lhd X$ and X is a soluble T-group, so $C_X(P/N) \cap C_X(X') \neq 1$; let x be a nonunit element of this intersection. Then $\langle x\rangle \lhd C_X(X') \lhd X$, so that $\langle x\rangle \lhd X$; consequently $[P, x] \lhd G$ and $C_P(x) \lhd G$. If $C_P(x) \neq 1$, then $N \leq C_P(x)$ and x belongs to $C_G(P/N) \cap C_G(N) = P$ by (22); thus $x \in P \cap X = 1$. Therefore $C_P(x) = 1$. Now x centralises P/N and consequently $[P, x] \leq N$. Evidently $[N, x] \lhd G$ and $[N, x] \neq 1$, which shows that $N = [N, x] = [P, x]$. Thus $[P, x] = [P, x, x]$. Let $a \in P$; then $[a, x] = [b, x, x]$ for some $b \in P$, and $C_P(x) = 1$ implies that $a = [b, x]$. Thus $P \leq [P, x] = N$. Consequently $N = P$ and G is of type VIII.

We are left with the following situation; there exists a nontrivial $M \lhd G$ with $M \cap N = 1$. If $M \cap P = 1$, then $M \leq C_G(P) = P$; therefore $M \cap P \neq 1$ and we can assume that $M \leq P$. Also $M \overset{G}{\simeq} MN/N$ shows that every subgroup of M is normal in G and we can assume M to have order p. Also N has order p since $N \overset{G}{\simeq} NM/M$. Suppose now that $P > M \times N$ and choose $a \in P \backslash (M \times N)$. The automorphism groups induced by G in P/M and P/N are both finite. Therefore $G/C_G(P)$ is finite, from which it follows that a^G is finite. Maschke' theorem implies that a^G is a direct product of minimal normal subgroups of G; therefore there exists a minimal normal subgroup L of G contained in P such that $L \not\leq M \times N$. Then $LM/M \overset{G}{\simeq} LN/N$. Let $g \in G$; then g induces in P/M and P/N power automorphisms which both have the form $a \longrightarrow a^n$ since they must agree on LM/M and LN/N. Hence $a^g = a^n$ for all $a \in P$, a situation we have already seen to be impossible (by Lemma 5.2.2 of [18]).

Hence $P = M \times N$ and X is isomorphic with a subgroup of GL$(2, p)$ which is diagonal because X induces power automorphism groups in M and N; this subgroup X is not scalar since it does not induce a group of power automorphisms in P. Clearly p is odd and G is of type IV.

(6.212) Case P **nilpotent of class 2.** Since P/N is abelian, $P' = N$. Consider the centraliser of P/N. If $g \in C_G(P/N)$, then $[g, P, P] = 1$ by (20); therefore, by the Three Subgroup Lemma, $[g, P'] = 1$, i.e. $[g, N] = 1$. Now let $a \in P$; then $a^g = ab$ where $b \in N$. Hence $a^{g^p} = ab^p = a$ since $[g, N] = 1$. It follows that $C_G(P/N)/C_G(P)$ is a p-group. But $C_G(P) \leq P$ by (21); therefore $C_G(P/N) \leq P$ and

$$C_G(P/N) = P. \tag{23}$$

Next, if $p = 2$, a periodic group of power automorphisms of P/N has order a power of 2 and equation (23) yields $P = G$, i.e. G is nilpotent. Thus p is an odd prime and therefore P is a regular p-group. Also N is the monolith of G; for suppose that $1 \neq M \lhd G$ and $M \cap N = 1$; then PM/M and PN/N are abelian, being Dedekind groups without elements of order 2, and therefore P is abelian, contrary to hypothesis.

Since $P \neq G$, we can find $g \in G \backslash P$ and (23) shows that g cannot centralise P/N. Let g induce in P/N the power automorphism $a \longrightarrow a^{\alpha}$; here α is a p-adic integer unit $\neq 1$. If $a, b \in P$, then $a^g \equiv a^{\alpha} \bmod N$ and $b^g \equiv b^{\alpha} \bmod N$; hence

$$[a, b]^g = [a^{\alpha}, b^{\alpha}] = [a, b]^{\alpha^2} \tag{24}$$

since $[N, P] = 1$. From this it follows that $\langle [a, b] \rangle \lhd G$. Since N is the monolith of G, the order of N is p; let $N = \langle a \rangle$, say.

Suppose that $P^p > 1$; then $N \leq P^p$ and consequently $a = b^p$ for some $b \in P$ since P is regular. Now $b^g = b^{\alpha}c$ for some $c \in N$, and $a^g = (b^g)^p = b^{\alpha p} = a^{\alpha}$. But equation (24) shows that $a^g = a^{\alpha^2}$; therefore $\alpha^2 \equiv \alpha \bmod p$ and $\alpha \equiv 1 \bmod p$. If P/N has finite exponent p^e, the congruence $\alpha^{p^{e-1}} \equiv 1 \bmod p^e$ implies that g induces in P/N an automorphism of order a power of p; therefore $g \in P$ by (23).

If, however, P/N has infinite exponent, α must have finite order; this, together with $\alpha \equiv 1 \bmod p$ and $p > 2$, implies that $\alpha = 1$. These arguments indicate that

$$P^p = 1. \tag{25}$$

Next the centre of P will be identified; call this Z. Clearly $N \leq Z$. Let $1 \neq a_1 \in Z$; then $\langle a_1, N\rangle \lhd G$, so $a_1^G \leq \langle a_1, N\rangle$, which implies that a_1^G is finite. Now $C_G(a_1^G) \geq P$; therefore Maschke's theorem can be applied to a_1^G; in the usual way it follows that $a_1^G = N$. Thus $Z = N$.

Now P/N is elementary abelian by (25); hence P is an extra-special p-group. Choose a basis for P/N, say $\{x_\lambda N \colon \lambda \in \Lambda\}$. Then

$$[x_\lambda, x_\mu] = a^{f(\lambda,\mu)} \tag{26}$$

where f is a nondegenerate alternating bilinear form. Now G/P is cyclic with order q dividing $p - 1$ since $P = C_G(P/N)$. Hence there is an element g with order q such that $G = P\langle g\rangle$ and $P \cap \langle g\rangle = 1$. Moreover $\langle g\rangle$ acts faithfully on P.

g induces in the elementary abelian group P/N a power automorphism of the form $x \longrightarrow x^n$ where $1 < n < p$. Thus

$$x_\lambda^g = x_\lambda^n a^{n_\lambda}, \quad (\lambda \in \Lambda), \tag{27}$$

for certain integers n_λ satisfying $0 \leq n_\lambda < p$. A suitable change of basis will simplify these equations. Suppose that $n_\lambda \neq 0$. Since f is nondegenerate, there is a $\mu \in \Lambda$ such that $f(\lambda, \mu) \not\equiv 0 \bmod p$. We shall replace x_λ by a suitable element of the form $\bar{x}_\lambda = x_\lambda^s x_\mu^t$. A brief computation using (26) and (27) yields $\bar{x}_\lambda^g = \bar{x}_\lambda^n a^u$ where $u = sn_\lambda + tn_\mu + st\binom{n}{2}\cdot f(\lambda, \mu)$. We wish to show that $u \equiv 0 \bmod p$ can be solved for s and t with $s \not\equiv 0 \bmod p$ and $t \not\equiv 0 \bmod p$. This amounts to solving

$$xn_\lambda + yn_\mu + z \equiv 0 \bmod p \tag{28}$$

for $x \not\equiv 0 \bmod p$ and $y \not\equiv 0 \bmod p$ where $z = \binom{n}{2}f(\lambda, \mu)$; notice that $z \not\equiv 0 \bmod p$. Since $n_\lambda \not\equiv 0 \bmod p$, we need only look for a y such that $yn_\mu + z \not\equiv 0 \bmod p$. If $n_\mu = 0$, any $y \not\equiv 0$ will do; if $n_\mu \neq 0$, we can choose y so that $1 \leq y < p$ and $yn_\mu + z \not\equiv 0 \bmod p$ since $p > 2$. Consequently (28) has a solution of the required sort.

Now replace x_λ by $\bar{x}_\lambda$, observing that we retain a basis for P/N. Performing this operation whenever necessary, we arrive at a basis for which $x_\lambda^g = x_\lambda^n$ for all $\lambda \in \Lambda$. Thus G is of type V.

(6.22) Case $[N, G'] = 1$ **and** P/N **nonabelian.** If a soluble p-group has the property T and is not abelian, then $p = 2$ [18, Lemma 4.2.1]. Thus P is a 2-group. Also (by Lemma 2.4.1 of [18]) L/N is a radicable abelian 2-group where, as usual, $L = [G', G]$. Define $C = C_G(G'/N)$ and note that C/N is nilpotent of class ≤ 2. Moreover G/C has order 1 or 2 because ± 1 are the only 2-adic integers with finite order.

Let $x \in C \cap P$; then $aN \to [a, x]$ is a homomorphism of L/N into N since $[N, C \cap P] = 1$ (see equation (19)). But $\mathrm{Hom}(L/N, N) = 0$; therefore

$$[L, C \cap P] = 1. \tag{29}$$

Since $L \leq G' \leq C \cap P$, it follows that L is abelian.

Our next aim is to prove that G is a 2-group. Since $|G:C| = 1$ or 2, all elements of G with odd order belong to C. Now C/N is a Dedekind group, so the elements in C/N which have odd order form an abelian subgroup, say Q/N. Assume that $Q \neq N$. From this it follows that $[N, Q] \neq 1$; for if $[N, Q] = 1$, the group Q is nilpotent, and, since $Q \lhd G$, we deduce from Lemma 5 that Q is a 2-group and $Q = N$. Now $[N, Q] \neq 1$ implies that $L = N$. For suppose $L > N$; then L/N is a nontrivial radicable abelian 2-group and $L^2 \neq 1$; therefore L/L^2 is radicable, which shows that L is radicable. Now commutation with a fixed element of C induces a homomorphism of L into N, and yet $\mathrm{Hom}(L, N) = 0$; thus $[L, C] = 1$ and in particular $[N, Q] = 1$, a contradiction. It follows that $L = N$ and G/N is Dedekind; thus $C = C_G(G'/N) = G$ and (29) becomes $[L, P] = 1$. Now $[N, Q] \neq 1$ implies that G splits over N, by Lemma 8; say $G = NX$ and $N \cap X = 1$. Therefore $P = P \cap (NX) = N(P \cap X)$. Since P/N is not abelian, $P \cap X \neq 1$. Also $[N, P \cap X] \leq [L, P] = 1$, so $P \cap X \lhd NX = G$. Thus $G/P \cap X$ is a T-group and the isomorphism $N \overset{G}{\simeq} N(P \cap X)/P \cap X$ shows that $|N| = 2$. Consequently $N \leq \zeta(G)$ which implies that G is nilpotent. This contradiction establishes that G is a 2-group. Equation (29) now yields

$$[L, C] = 1 \quad \text{and} \quad C = C_G(L). \tag{30}$$

Observe that G/N is not a Dedekind group; for if it were, $L = N$ and $C = G$, so that (30) would become $[L, G] = 1$, i.e. G is nilpotent. Also L is radicable; for $L > N$ and this, as has already been seen, implies that L is radicable.

G/N is a soluble 2-group with the property T and it is also nonnilpotent. By Lemma 4.2.1 of [18] this means that one can write $G = \langle C, t\rangle$ where t transforms each element of C/N into its inverse and $t^2 \in C$; also, of course, C/N is abelian and of infinite exponent. Equation (30), together with the commutativity of C/N, implies that C *is nilpotent of class at most* 2. Let σ be the automorphism $a \to a^{-1}$ of L and write τ for the automorphism of L induced by t. Then $\tau^{-1}\sigma$ is trivial on L/N and $\tau^{-1}\sigma - 1 \in \mathrm{Hom}(L, N) = 0$. Therefore $\tau = \sigma$ and

$$a^t = a^{-1}, \quad (a \in L), \tag{31}$$

which shows that t centralises N. Since $G = \langle C, t\rangle$, equation (30) permits us to conclude that $[N, G] = 1$.

Suppose there exists $M \lhd G$ with $M \neq 1$ and $M \cap N = 1$. Since $M \overset{G}{\simeq} MN/N$, one can assume that M has order 2. Also $L \nleq M$, so G/M is not a Dedekind group and its structure is similar to that of G/N. In particular CM/M—and hence C—is abelian. Now t transforms elements of L into their inverses. It follows that $a^t =$

a^{-1} for all $a \in C$, which is impossible by Lemma 6. Hence G is monolithic with monolith equal to N. This indicates that L is of type 2^∞ and N has order 2.

Define Z to be the centre of C. Then $L \leq Z$ and, L being radicable, we may write $Z = L \times D$. Suppose that D contains an element d of order 4; then $d^t = d^{-1}a$ for some a in N. Hence $(d^2)^t = (d^{-1}a)^2 = d^{-2} = d^2$. Since $d^2 \in Z$, it follows that d^2 is in the centre of G and $1 \neq \langle d^2 \rangle \lhd G$; this is impossible since $\langle d^2 \rangle \cap N = 1$. Thus D is elementary abelian. This implies that DN/N lies in the centre of G/N, so that

(32) $$[D, G] \leq N.$$

If $d \in D$, the mapping $xC \longrightarrow [x, d]$ is a homomorphism of G/C into N since $[C, D] = 1 = [N, G]$. Now $C_D(G) \lhd G$, so $C_D(G) = 1$, and, since $\mathrm{Hom}(G/C, N)$ has order 2, we must have $|D| = 1$ or 2. Write $D = \langle d \rangle$.

Consider next the position of t^2. Let $a_1, a_2, \cdots$ be a canonical set of generators for the 2^∞-group L. If $c \in C$, then $c^t = c^{-1}a$ for some a in N. Hence $c^{t^2} = (c^{-1}a)^{-1}a = c$. On account of $G = \langle C, t \rangle$ it follows that $t^2 \in \zeta(G)$; in particular $\langle t^2 \rangle \lhd G$. Therefore, either $t^2 = 1$ or $N \leq \langle t^2 \rangle$. Also $t^2 \in Z$ and since t^2N is centralised by t, the element t^2 has order dividing 4. Thus $t^2 \in \langle a_2 \rangle \times \langle d \rangle$ and the possibilities for t^2 are 1, a_1 or a_2d (if $d \neq 1$); for if $d \neq 1$, then $d^t = a_1d$ by (32) since $\langle d \rangle$ cannot be normal in G.

(6.221) **Case C abelian.** Here $C = Z = L \times D$, and $d \neq 1$ by Lemma 6. Since $d^t = a_1d$, we have $(td)^2 = t^2a_1d^2 = t^2a_1$. Hence $t^2 = a_1$ implies that $(td)^2 = 1$. Therefore we can assume that either $t^2 = 1$ or $t^2 = a_2d$; the order of t is 2 or 8. Thus G is of type II.

(6.222) **Case C nilpotent of class 2.** Since C/N is abelian, $C' = N \leq Z$. If x and y belong to C, then $1 = [x, y]^2 = [x^2, y]$, showing that C/Z is elementary abelian. Choose a basis for C/Z, say $\{x_\lambda Z : \lambda \in \Lambda\}$; then $x_\lambda^2 = ad^i$ where $a \in L$ and $i = 0$ or 1. Now $a = b^2$ for some $b \in L$ and $(x_\lambda b^{-1})^2 = x_\lambda^2 a^{-1} = d^i$. Write $\bar{x}_\lambda = x_\lambda b^{-1}$; then $\bar{x}_\lambda^t = \bar{x}_\lambda^{-1}c$ for some c in N and $(\bar{x}_\lambda^2)^t = (\bar{x}_\lambda^{-1}c)^2 = \bar{x}_\lambda^{-2}$. It follows that $\langle d^i \rangle \lhd G$, which can only mean that $d^i = 1$ and $\bar{x}_\lambda^2 = 1$. In short, we can assume that

(33) $$x_\lambda^2 = 1$$

for all $\lambda \in \Lambda$.

Define $X = \langle x_\lambda : \lambda \in \Lambda \rangle$. From (33) it follows that $X^2 = X'$; also $C = XZ$, so $N = C' = X'$ and $N = X' = X^2$. Suppose that $u \in X \cap Z$ and write $u = x_{\lambda_1}^{n_1} \cdots x_{\lambda_r}^{n_r} a$ where $a \in N$, the n_i are integers and the λ_i are distinct elements of Λ. The independence of the $x_{\lambda_i} Z$ indicates that each n_i is even; thus $u \in X^2N = N$. Consequently

(34) $$X \cap Z = N.$$

Therefore $\zeta(X) \leq X \cap \zeta(C) = X \cap Z = N$, and $\zeta(X) = N$. Also, $X/N \simeq C/Z$, an elementary abelian 2-group. We conclude that X is an extra-special 2-group generated by elements of order 2. Clearly the group C is a direct product of X and Z in which $\zeta(X)$ and $\langle a_1 \rangle$ are amalgamated.

It has been remarked that $t^2 = 1$, a_1 or $a_2 d$ (if $d \neq 1$); in fact the second possibility can be discarded if t is chosen suitably. The argument for this has already been given in the last part of (5.332).

Since t acts trivially on both X/N and N, the map $\sigma: xN \longrightarrow [x, t]$ is an element of $\mathrm{Hom}(X/N, N)$. If $d \neq 1$, one can assume that $\sigma = 0$ and $[X, t] = 1$. For in this case if $x_\lambda^t = x_\lambda a_1$, we obtain $(x_\lambda d)^t = x_\lambda d$ while $(x_\lambda d)^2 = 1$. Thus G is of type III.

[In conclusion, observe that even if $d = 1$ one can still take $\sigma = 0$ at the expense of losing $x_\lambda^2 = 1$; for $(x_\lambda a_2)^t = x_\lambda a_2$ if $x_\lambda^t \neq x_\lambda$.]

7. **Proof of Theorem 1 concluded.** It remains to show that a group G of types I to IX is a JNT-group; in each case G is obviously soluble. One first observes that in no case is G a T-group. For types II, IV and VIII this is clear. For types I, III, V and VII it follows from the structure of Dedekind groups. If G is of type VI(a), then D is a nonnormal subnormal subgroup, as is $\langle u \rangle$ if G is of type VI(b). If G is of type IX, then 1_F generates a nonnormal subnormal subgroup since $X \neq \langle -1_F \rangle$.

Next it must be shown that every proper factor group of G is a T-group. If G is of type I or VIII this is clear. All types except IV and IX are monolithic and if N is the monolith one merely has to verify that G/N is a T-group. If G is of type II or III, then $N = \langle a_1 \rangle$ and $G/\langle a_1 \rangle$ fits the prescription for a soluble 2-group with the property T (see [18, Theorem 3.1.1]). If G is of type V, then $N = \zeta(P)$ and G/N is a T-group by Lemma 5.2.2 of [18]. If G is of type VI or VII, then $N = \langle a_1 \rangle$ and $H = G/N$ has a normal p^∞-subgroup K such that H/K is abelian and all subgroups of $C_H(K)$ are normal in H; a subnormal subgroup S of H either contains K or lies in $C_H(K)$; hence $S \lhd H$.

Turning to the nonmonolithic groups, we see that in type IV a nontrivial normal subgroup N of G contains one of the two normal subgroups of order p; hence G/N is a T-group by Lemma 5.2.2 of [18].

This leaves us with the case where G is of type IX. Let $1 \neq N \lhd G$; then certainly $N \cap F$ is nontrivial. Let $0 \neq f \in N \cap F$. Since $F = Q + X^+$, we can write

$$f^{-1} = \sum_{x \in X} r_x x, \quad (r_x \in Q).$$

Choose a positive integer n such that each nr_x is integral and observe that N contains the element

$$\sum_{x \in X} (nr_x) xf = nf^{-1} f = n.$$

Hence $nX^+ \leq N$. We shall show that G/nX^+ is a T-group. $F = Q + X^+$ is divisible as an additive abelian group and $F/(Q + nX^+)$ has finite exponent. Thus $F = Q + nX^+$ and F/nX^+ is isomorphic with a factor group of $Q/\langle 1 \rangle$. Hence every automorphism of F/nX^+ is a power automorphism and every subgroup of F/nX^+ is normal in G/nX^+. If $x \neq 1$, then $F(x - 1) = F$, which is easily seen to imply that a subnormal subgroup of G/nX^+ either contains F/nX^+ or is contained in it. This shows that G/nX^+ is a T-group.

8. Finitely generated soluble JNT-groups.

The main result of this section is

Theorem 2. *A finitely generated hyperabelian group which is not a T-group has a finite homomorphic image which is not a T-group.*

Recall here that a group is *hyperabelian* if it possesses an ascending series of normal subgroups whose factors are all abelian; this is equivalent to requiring each nontrivial homomorphic image to have a nontrivial normal abelian subgroup.

Proof. Let G be a finitely generated hyperabelian group which is not a T-group. Suppose that $\{N_\alpha: \alpha \in A\}$ is a chain of normal subgroups of G such that no G/N_α is a T-group; write N for the union of the chain. Assume that G/N is nevertheless a T-group. Now a hyperabelian T-group is soluble because soluble T-groups are metabelian; moreover, a finitely generated soluble T-group is either finite or abelian [18, Theorem 3.3.1], and therefore is certainly finitely presented. Thus G/N is finitely presented. By a well-known principle this implies that $N = \langle a_1, \cdots, a_n \rangle^G$ for a certain finite set of a_i's. Hence $N = N_\alpha$ for some α. By this contradiction G/N is not a T-group. Zorn's Lemma shows that there exists a normal subgroup M of G which is maximal subject to G/M not being a T-group. Clearly G/M is a JNT-group. Moreover G/M contains a nontrivial normal abelian subgroup, being hyperabelian. Thus G/M is soluble and Theorem 2 will follow from

Lemma 9. *A finitely generated soluble JNT-group is finite* (*and hence of type* I, IV, V *or* VIII).

Proof. One can, of course, verify directly that no soluble JNT-group on our list can be both finitely generated and infinite. However, it is more economical to proceed independently as follows.

Let G be a finitely generated soluble JNT-group which is infinite. First of all observe that G cannot be nilpotent; for if G were nilpotent, the initial argument of §3 would show that G is periodic and this, as is well known, implies that G is finite.

Denote by A a nontrivial normal abelian subgroup of G. Suppose that G/A is infinite. If $1 < B \leq A$ and $B \lhd G$, then G/B is abelian and $G' \leq B$. Hence G' is minimal normal in G and lies in A. Therefore $G/C_G(G')$ is a finitely generated abelian group and by a theorem of P. Hall [8, Theorem 5.1], G' is a finite elementary abelian p-group for some prime p.

Now write $C = C_G(G')$; then G/C is finite. Also $C' \leq G' \leq \zeta(C)$; thus if $x, y \in C$, we have $1 = [x, y]^p = [x^p, y]$. Hence $C^p \leq \zeta(C)$ and C^p is abelian. Now G/C^p is periodic and hence finite, so C^p is finitely generated and infinite. Hence for some integer n the group $N = (C^p)^n$ is torsion-free and nontrivial, while G/N is finite. This contradicts Lemma 5.

Therefore G/A is finite, which shows that A is finitely generated and infinite; there is no loss in assuming A to be free abelian. Let $L = [G', G]$ and observe that $L \neq 1$. If $L \cap A \neq 1$, then $G/L \cap A$ is finite, by the first part of the proof, and, replacing A by $L \cap A$, we may assume that $A \leq L$. Let p be a prime dividing $|G : L|$; then G/A^p is a finite soluble T-group; however the prime p divides both $|G:L|$ and $|L : A^p|$, which is impossible [7]. Thus $L \cap A = 1$ and $L \simeq LA/A$, which shows that L is finite and abelian; therefore G/L is infinite. However this situation has been shown to be impossible.

Lemma 9 may be compared with B. H. Neumann's theorem that a finitely generated soluble just nonabelian group is finite [15, Theorem 6.3]—see also Rosati [21]. It is not difficult to show that a soluble just nonabelian group cannot be a T-group if it is infinite. Thus Neumann's theorem is a special case of Lemma 9.

9. $JN\overline{T}$**-groups.** A group G has the property $\overline{T}$ if $H \lhd K \lhd L \leq G$ always implies that $H \lhd L$. Thus $\overline{T}$-groups form the largest subgroup-closed subclass of the class of T-groups.

A $JN\overline{T}$-group is either a JNT-group or a T-group. There exist finite $JN\overline{T}$-groups which are T-groups, for example the symmetric group S_n where $n \geq 5$; but this phenomenon cannot occur in the soluble case. In fact we shall prove

Theorem 3. *A group is a soluble* $JN\overline{T}$*-group if and only if it is isomorphic with a group of type* I, IV, V *or* VIII *(with* X *a* $\overline{T}$*-group in the last case).*

Proof. Let G be a soluble $JN\overline{T}$-group. First observe that G is not a T-group; for suppose this is wrong. If G is a T-group of type I, then $L = [G', G]$ contains an element a of infinite order; therefore $L/\langle a^4\rangle$ has an element of order 4 and $G/\langle a^4\rangle$ is not abelian, which precludes $G/\langle a^4\rangle$ from being a $\overline{T}$-group (for this and other results about soluble $\overline{T}$-groups see [18, Theorem 6.1.1]). If G is a T-group of type II, then G' is radicable and if $1 \neq a \in G'$, the group $G/\langle a\rangle$ is also a T-group of type II and therefore not a $\overline{T}$-group. Finally, if G is periodic, L has an element of order 2 since G would otherwise be a $\overline{T}$-group. Therefore there is a normal subgroup $N \neq 1$ of G such that L/N is of type 2^∞; however this prevents G/N from being a $\overline{T}$-group.

Therefore G is a JNT-group and appears on our list. It remains to discard those JNT-groups G which have a factor group G/N such that $N \neq 1$ and G/N

is not a $\overline{T}$-group. Here one must keep in mind that a soluble $\overline{T}$-group H is either periodic or abelian and that $[H', H]$ cannot have an element of order 2. This excludes types II, III, VI, VII and IX—recall that in type IX the group X cannot be periodic. There is no difficulty in verifying that the remaining types are $JN\overline{T}$-groups (in type VIII one must assume that X is a soluble $\overline{T}$-group).

Locally finite $JN\overline{T}$-groups. A good deal more is known about $\overline{T}$-groups than T-groups and one would hope for correspondingly more information about $JN\overline{T}$-groups. For example, *a locally finite $\overline{T}$-group is soluble and therefore metabelian.* This is an easy corollary of the well-known theorem of Huppert that a finite group with all of its proper subgroups supersoluble is soluble [12, Satz 22]; see also [19].

We shall outline a method of describing the locally finite $JN\overline{T}$-groups that are insoluble; this is based on the Fitting-Gol'berg theory of semisimple groups (see [14, §61). Let G be a locally finite $JN\overline{T}$-group which is insoluble. Then every proper factor group of G is metabelian and $M = G''$ is the monolith of G. Then G acts *irreducibly* on M, i.e., M has no proper nontrivial subgroups that are G-admissible; in particular, M is characteristically simple. Clearly M is perfect and its centre is 1. Thus $C_G(M) = 1$, from which it follows that there is an isomorphism of G with a subgroup of Aut M, the full automorphism group of M, in which M is mapped onto the group of inner automorphisms Inn M. Thus one may assume that

$$\text{Inn } M < G \leq \text{Aut } M.$$

Conversely, let M be a nonabelian, characteristically simple group which is locally finite; let G be a subgroup of Aut M which contains Inn M and acts irreducibly on Inn M; assume also that $G/\text{Inn } M$ is a $\overline{T}$-group. Then in fact G is a $JN\overline{T}$-group. To prove this let $1 \neq N \lhd G$; if $N \cap (\text{Inn } M) = 1$, then $[N, \text{Inn } M] = 1$ and $[N, M] \leq \zeta(M) = 1$, which shows that $N = 1$. Thus Inn $M \leq N$ by irreducibility and G/N is a $\overline{T}$-group. On the other hand G is not a $\overline{T}$-group since if it were, Inn M—and therefore M—would be soluble. Also Inn M is the monolith of G.

Suppose that G_1 and G_2 are two isomorphic groups obtained in this way from groups M_1 and M_2; then evidently $M_1 \simeq M_2$. If we identify M_1 and M_2 and write α for the isomorphism of G_1 with G_2, then α determines by restriction an automorphism α^* of M_1. It is routine to check that $g^{\alpha} = (\alpha^*)^{-1} g^{\alpha^*}$, $(g \in G_1)$. This is summed up in

Theorem 4. *There is a one-one correspondence between isomorphism classes of insoluble, locally finite, $JN\overline{T}$-groups with given monolith M and conjugacy classes of irreducible, locally finite $\overline{T}$-subgroups of* Out M (*the group of outer automorphisms of* M).

Of course it is by no means clear which groups M can arise here. However one obvious candidate—and the only one if M is finite or merely possesses a minimal normal subgroup—is a direct power of a nonabelian simple group H. For simplicity of presentation suppose we are dealing with *finite* $J N\overline{T}$-groups. Let M be the direct product of n copies of H. Then, as was shown by Fitting [5, Satz 12], $\operatorname{Aut} M \simeq (\operatorname{Aut} H) \wr S_n$ and $\operatorname{Out} M \simeq (\operatorname{Out} H) \wr S_n$; in these wreath products Aut H and Out H are in their regular representations and the symmetric group S_n is in its natural permutation representation. The irreducible subgroups of Out M correspond in the isomorphism to subgroups of $(\operatorname{Out} H) \wr S_n$ which map onto transitive subgroups of S_n in the canonical homomorphism of $(\operatorname{Out} H) \wr S_n$ onto S_n.

Thus *a finite* $J N\overline{T}$*-group is either soluble—and therefore of type* I, IV, V or VIII—*or insoluble, in which case it corresponds to a conjugacy class of irreducible* $\overline{T}$*-subgroups of* $(\operatorname{Out} H) \wr S_n$ *where* H *is a finite nonabelian simple group and* n *a positive integer* > 1. However the problem of adequately describing all finite JNT-groups remains open.

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